General Relativistic Stars: Linear Equations of State

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In this paper Einstein's field equations, for static spherically symmetric perfect fluid models with a linear barotropic equation of state, are recast into a 3-dimensional *regular* system of ordinary differential equations on a compact state space. The system is analyzed qualitatively, using the theory of dynamical systems, and numerically. It is shown that certain special solutions play important roles as building blocks for the solution structure in general. In particular, these special solutions determine many of the features exhibited by solutions with a regular center and large central pressure. It is also shown that the present approach can be applied to more general classes of barotropic equations of state.

Key Words: static spherical symmetry; stellar models; linear equation of state

1. INTRODUCTION

In both Newtonian gravity and general relativity, the simplest models of isolated stars are given by static spherically symmetric configurations. Despite their simplicity, they are believed to yield many insights about much wider classes of stellar models (see, for example, the discussion by Hartle [6]).

The line element for a static spherically symmetric model can be written as

$$ds^2 = -e^{2\phi(\lambda)}dt^2 + r(\lambda)^2\left[\tilde{N}(\lambda)^2 d\lambda^2 + d\Omega^2\right] , \tag{1}$$

with

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2 , \tag{2}$$

where $\phi(\lambda)$ is the gravitational potential, $\tilde{N}(\lambda)$ a dimensionless (under scale-transformations) freely specifiable function, and $r(\lambda)$ the usual Schwarzschild radial parameter, associated with the area of the spherical symmetry surfaces. The coordinate λ is a spatial radial variable, whose interpretation depends on the choice of $\tilde{N}$. Since $\tilde{N} = 1$ corresponds to isotropic coordinates, the function $\tilde{N}$ can be viewed as a relative gauge function with respect to the isotropic gauge.

This paper is the first in a series devoted to the study of general relativistic star models with perfect fluid sources. Thus the energy-momentum tensor is assumed to be of the form

$$T_{ab} = \rho u_a u_b + p\left(g_{ab} + u_a u_b\right) , \tag{3}$$

where ρ is the energy density, p the pressure, and u^a the 4-velocity of the fluid. A relation between the gravitational potential ϕ in (1) and the matter content can be found from the equations of motions for the fluid, $\nabla_a T^{ab} = 0$, namely

$$\frac{d\phi}{dp} = -\frac{1}{\rho + p} . \tag{4}$$

This equation is the general relativistic generalization of the corresponding Newtonian equation $d\phi/dp = -1/\rho$.

For a static spherically symmetric perfect fluid model to be considered as describing a star, we require that it is isolated in the sense that the model has a boundary at a finite radius where the pressure of the fluid vanishes. At this radius, the interior solution is matched with a static

exterior vacuum spacetime, described by the Schwarzschild solution

$$ds^2 = -\left(1 - \tfrac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \tfrac{2M}{r}\right)} + r^2 d\Omega^2 \ , \tag{5}$$

(see Schwarzschild [12]). A vanishing pressure is, in fact, a necessary and sufficient matching condition (see, for example, Stephani [13], p 161).

In order to analyze the gravitational field equations for (1), an equation of state must be specified. Over the years, a number of exact static spherically symmetric perfect fluid solutions have been found, see, for example, Delgaty & Lake [4] for a comprehensive review. Unfortunately, most equations of state for known exact solutions have no physical motivation. They are instead chosen solely with the purpose of simplifying the differential equations, and thereby allowing exact solutions to be found. Our aim is not to find new exact solutions, but to gain an understanding of the solution space, and its implications, for given equations of state which can be physically motivated.

We will consider barotropic equations of state

$$\rho = \rho(p) \ , \tag{6}$$

which dominate in the literature (see Stergioulas [14] and references therein). Such equations of state are relevant for describing neutron stars and white dwarfs, see, for example, Misner *et al* [2], p 624. As a first step we will consider complete analytic equations of state covering the pressure range of the entire star models. Once the solution space for such equations of states are understood, models with composite equations of state (*i.e.* models where the equations of state are obtained by matching different equations of state in different pressure regions) can be obtained by matching different matter solutions (or in the terminology introduced below, by matching orbits in different state spaces).

One might think that static spherically symmetric perfect fluid models lead to simple problems and that there is not much to be discovered in this area of research. In this series of papers we will show that this is not the case. Some problems turn out to be quite complicated and one can obtain new insights.

The most favored approach for studying static stars, the so-called Tolman–Oppenheimer–Volkoff approach, uses the pressure and mass as dependent variables and the Schwarzschild radial coordinate as the independent variable (see, for example, section 6.2 in Wald [18]). The resulting equations, however, are not regular at the center of the star, and the problem of proving existence and uniqueness of solutions requires a quite technical analysis of singular differential equations, see Rendall & Schmidt [11].

Our method is to recast the field equations for a given equation of state as a regular 3-dimensional system of ordinary differential equations on a compact state space. This *regularization* means that the problems associated with the singular nature of the Tolman–Oppenheimer–Volkoff equations are circumvented. In addition, recasting the field equations into regularized form on a compact state space allows us to conveniently investigate the solution space using powerful methods from the theory of dynamical systems. Such methods have been used in spatially homogeneous cosmology with great success, see Wainwright & Ellis [17] and references therein. The regularized form is also suitable for numerical calculations. The fact that the dynamical system is compact and 3-dimensional is of great advantage, since this makes it possible to *visualize* the state space and thereby obtain a clear picture of the structure of the *entire* solution space.

We will show that there exist special solutions that play important roles as building blocks for the remaining solution structure. Some of these solutions turn out to determine many of the features exhibited by solutions with a regular center and a large central pressure. We will also show that the behavior of the solutions at small (and to some extent also large) radii is intimately connected with asymptotic self-similarity, even for non-scale-invariant equations of state. A useful formulation for the gravitational field equations of the static spherically symmetric perfect fluid models and a good understanding of the corresponding solution structure may also serve as a starting point for exploring their role in a broader context, as will be discussed in the concluding remarks.

In this paper we consider the linear equation of state

$$\rho = \rho_0 + (\eta - 1)p \ , \tag{7}$$

where the constants ρ_0 and η satisfy $\rho_0 \geq 0, \eta \geq 1$. The case $\eta = 1$ corresponds to an incompressible fluid with constant energy density, while the case $\rho_0 = 0$ describes a scale-invariant equation of state. The scale-invariant case has been investigated previously by Collins [3] using dynamical systems theory. Note that $\eta < 2$ corresponds to non-causal fluids, in which the velocity of sound is greater than that of light.

The outline of the paper is as follows: in Section 2 we write the gravitational field equations as a 3-dimensional regular dynamical system on a compact state space and perform a local analysis of the resulting equations. We also present a monotone function and describe the dynamical behavior on the boundaries. In Section 3 we focus on the regular solutions, and discuss their behavior. In Section 4 we discuss how one can adapt the approach of this paper to facilitate studies of more general barotropic equations of state. We conclude with some remarks in Section 5.

Throughout the paper, geometric units with $c = G = 1$ are used, where c is the speed of light and G the gravitational constant. Roman indices, $a, b, ... = 0, 1, 2, 3$ denote spacetime indices.

2. THE DYNAMICAL SYSTEM FORMULATION

In order to recast the gravitational field equations for static spherically symmetric prefect fluid models with a linear equation of state into a regular dynamical system on a compact state space, we proceed as follows. We first write the line element in the form

$$ds^2 = -\mathrm{e}^{2\phi}dt^2 + d\ell^2 + e^{2\psi-2\phi}d\Omega^2 \ . \tag{8}$$

Then we introduce the variables

$$\theta = \dot{\psi} \ , \quad \sigma = \dot{\theta} \ , \quad B = e^{\phi-\psi} \ , \tag{9}$$

where a dot denotes differentiation with respect to ℓ. The gravitational field equations, expressed in these variables, are

$$\begin{aligned}
\dot{\theta} &= -2\theta^2 + \theta\sigma + B^2 + 16\pi p \ , && \text{(10a)} \\
\dot{\sigma} &= -2\theta\sigma + \sigma^2 + 4\pi(\rho + 3p) \ , && \text{(10b)} \\
\dot{B} &= (-\theta + \sigma)B \ , && \text{(10c)} \\
8\pi p &= \theta^2 - \sigma^2 - B^2 \ . && \text{(10d)}
\end{aligned}$$

This system is very similar to those obtained in spatially homogeneous cosmology (see Wainwright & Ellis [17]), even though the physical interpretation is quite different. One can hence import ideas from treatments of spatially homogeneous models to the present context.

To obtain a compact state space we introduce the variables

$$\{Q \ , S \ , C\} \ , \tag{11}$$

according to

$$\begin{aligned}
Q &= \frac{\theta}{\sqrt{\theta^2 + 8\pi\rho_0/\eta}} \ , \quad S = \frac{\sigma}{\sqrt{\theta^2 + 8\pi\rho_0/\eta}} \ , \\
C &= \frac{B^2}{\theta^2 + 8\pi\rho_0/\eta} \ .
\end{aligned} \tag{12}$$

These variables are closely related to those used by Uggla *et al* [16] in cosmology and by Nilsson *et al* [10] for studying static cylinders with a

linear equation of state. We also introduce and a new independent variable λ, defined by

$$\frac{d\ell}{d\lambda} = \frac{1}{\sqrt{\theta^2 + 8\pi\rho_0/\eta}} \ . \tag{13}$$

The above choices correspond to

$$\tilde{N}^2 = C \ , \quad \phi' = S \ , \quad r^2 = \frac{\eta(1-Q^2)}{8\pi\rho_0 C} \ , \tag{14}$$

in (1), and the prime denotes differentiation with respect to the independent spatial variable λ. From (14) it follows that C and $1-Q^2$ are positive. Integrating (4), yields

$$e^{\phi} = \alpha \left(\frac{1-Q^2}{1-S^2-C} \right)^{1/\eta} \ , \tag{15}$$

where α is a freely specifiable constant corresponding to the freedom of scaling the time coordinate t in the line element (1). This, in turn, reflects the freedom in specifying the value of the gravitational potential ϕ at some particular value of r. Matching an interior solution with the exterior Schwarzschild solution, however, fixes this constant. For the purpose of interpreting the variables Q and S, it is worth noting the relation

$$\frac{d\phi}{d\ln r} = \frac{S}{Q-S} \ . \tag{16}$$

In terms of the new variables, the gravitational field equations (10a)-(10c) takes the form

$$Q' = (1-Q^2)\left(QS - C - 2S^2\right) \ , \tag{17a}$$
$$S' = \tfrac{1}{2}W\left(2+\eta-2QS\right) - (1-S^2)\left(1-Q^2+QS\right) \ , \tag{17b}$$
$$C' = 2\left[S(1-Q^2) + Q\left(S^2 - W\right)\right] C \ . \tag{17c}$$

Equation (10d) leads to

$$W = 1 - S^2 - C \ , \tag{18}$$

where W is defined by

$$W = 8\pi\eta^{-1} C r^2 (p+\rho) \ , \tag{19}$$

and satisfies $W \geq 0$ if we assume that the weak energy condition $p + \rho \geq 0$ is satisfied. Equation (18) is used to eliminate W in (17a)-(17c). It follows from (18), (17b), and (17c), that

$$W' = \left[4QS^2 - 2SQ^2 - \eta S + 2QC\right] W \ . \tag{20}$$

Hence, $W = 0$ is an invariant subset, as are $Q = \pm 1$ and $C = 0$, which is easily seen from (17a) and (17c). These invariant subsets constitute the boundary of the physical state space for spherically symmetric models with a non-scale-invariant linear equation of state, *i.e.*, they describe the boundary of the part of state space where $C > 0$, $W > 0$ and $\rho_0 \neq 0$. We now include these boundaries and obtain a regular dynamical system (17a)-(17c) on a compact state space.

If the variables (11) are used for plane-symmetric models,[1] (17c) decouples and the remaining equations are identical to the $C = 0$ subset of (17a)-(17c). It is therefore natural to refer to this boundary as the plane-symmetric boundary. The $Q = \pm 1$ subsets correspond to setting $\rho_0 = 0$ in (7), and the remaining equations describe models with a scale-invariant equation of state $\rho = (\eta - 1)p$.[2] We refer to these boundaries as the scale-invariant boundaries. The state space, with the different boundaries identified, is shown in figure 1.

An important relation is

$$\frac{\eta p}{\rho_0} = \frac{Q^2 - S^2 - C}{1 - Q^2} \ , \tag{21}$$

since it yields

$$Q^2 - S^2 - C \geq 0 \ , \tag{22}$$

whenever the pressure is non-negative. The expression $Q^2 - S^2 - C = 0$ defines a surface in state space, which we refer to as the surface of vanishing pressure. This surface is also shown in figure 1. It is not an invariant subset of (17a)-(17c).

A monotone function excludes equilibrium points, periodic orbits, recurrent orbits, and homoclinic orbits in its domain. The function

$$Z = \frac{2Q - S}{\sqrt{(2Q - S)^2 + 3(1 - Q^2)}} \ , \tag{23}$$

[1] These models have the same line element as that in (1) but $d\Omega^2 = dx^2 + dy^2$.

[2] To obtain the line element for this case, an additional dimensional variable needs to be considered as well.

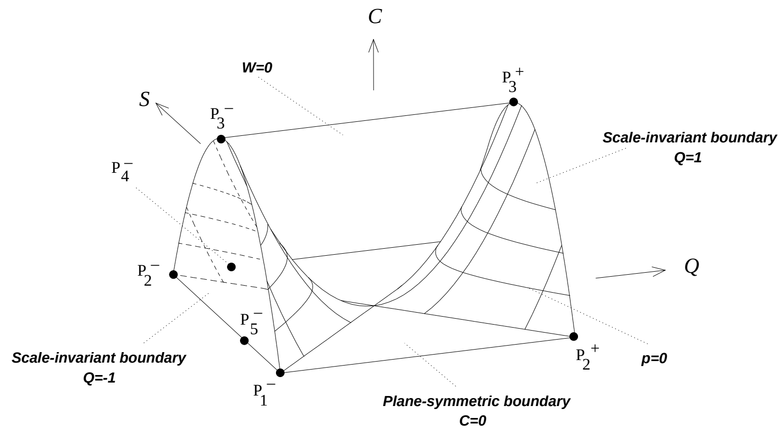

FIG. 1. The state space for static spherically symmetric models with a linear equation of state. The different boundaries are identified along with the surface of vanishing pressure, $p = 0$.

which satisfies

$$Z' = -\frac{3(1-Q^2)\left[2C + \eta W + 2(2S-Q)^2\right]}{2\left[(2Q-S)^2 + 3(1-Q^2)\right]^{3/2}} \,, \tag{24}$$

is monotonically decreasing and is defined everywhere in state space except on the scale-invariant boundaries $Q = \pm 1$. Hence, the "past" $(r \to 0)$ and "future" $(r \to +\infty)$ attractors lie on the $Q = \pm 1$ boundary subsets. On these boundaries, however, there exist other monotone functions which imply that the only attractors are equilibrium points on the $Q = \pm 1$ boundaries.[3] The equilibrium points of (17a)-(17c), together with their corresponding eigenvalues are listed in Table 1. The equilibrium points describe self-similar exact solutions to the gravitational field equations. The points $P^{\pm}_{1,2}$ correspond to the plane-symmetric vacuum solutions, first found by Kasner [7]. The points $P^{\pm}_3$ correspond to the Minkowski spacetime on explicitly spherically symmetric form. The points $P^{\pm}_4$ correspond to a non-regular self-similar perfect fluid solution due to Tolman [15]. This solution is, however, also associated with many other authors, for example, Misner & Zapolsky [9]. The points $P^{\pm}_4$ change from focuses to nodes at $\eta = \sqrt{64/7} - 2$. The equilibrium points $P^{\pm}_5$ correspond to a self-similar plane-symmetric perfect fluid model. Since this model only exists for $\eta < 2$,

[3]The equations on the boundary subsets $Q = \pm 1$ describe models with a scale-invariant equation of state. These equations exhibit a monotone function given in Goliath *et al* [5].

the fluid is necessarily non-causal. There is thus a bifurcation associated with the transition from a non-causal to a causal fluid at $\eta = 2$. The equilibrium points $P_5^{\pm}$ leave the state space through the points $P_1^{\pm}$ respectively, thereby changing these points from nodes to saddles.

TABLE 1.
Equilibrium points and their stability for the linear equation of state using the variables $\{Q, S, C\}$. The constant a is given by $a = \eta^2 + 4\eta - 4$. The points $P_5^{\pm}$ only exist for $1 \leq \eta < 2$. The constant ϵ takes the discrete values ± 1.

Eq point	Q	S	C	Eigenvalues
$P_1^{\pm}$	± 1	± 1	0	$\pm(2-\eta)\ ,\ \pm 2\ ,\ \pm 2$
$P_2^{\pm}$	± 1	∓ 1	0	$\pm(6+\eta)\ ,\ \pm 6\ ,\ \pm 2$
$P_3^{\pm}$	± 1	0	1	$\pm 2\ ,\ \mp 1\ ,\ \pm 2$
$P_4^{\pm}$	± 1	$\pm\frac{2}{2+\eta}$	$\frac{a}{(2+\eta)^2}$	$\mp\frac{1}{2+\eta}\left[2+\eta+\epsilon\sqrt{(2+\eta)^2-8a}\right]\ ,\ \pm\frac{2\eta}{2+\eta}$
$P_5^{\pm}$	± 1	$\pm\frac{1}{4}(2+\eta)$	0	$\pm\frac{1}{4}(2+\eta)\ ,\ \mp\frac{1}{8}(2-\eta)(6+\eta)\ ,\ \pm\frac{1}{4}a$

The line element expressed in the variables (11) is invariant under the discrete symmetry

$$(Q, S, \lambda) \to -(Q, S, \lambda)\ , \tag{25}$$

and since (17a)-(17c) are also invariant under this discrete symmetry, a solution is represented by two orbits in the state space. We can, however, without loss of generality, focus on orbits entering the state space from the $Q = 1$ boundary subset.

We note that all orbits which correspond to solutions with $\rho_0 \neq 0$, with spherical as well as plane symmetry, start at equilibrium points on the $Q = 1$ boundary and end at equilibrium points on the $Q = -1$ boundary. All solutions are thus asymptotically self-similar. However, on their way to $Q = -1$ from $Q = 1$ they all intersect the surface of vanishing pressure, $Q^2 - S^2 - C = 0$, at an interior point (Q_*, S_*, C_*) of the state space. To obtain physically reasonable spherically symmetric models (models with non-negative pressure) one matches each interior solution with the exterior Schwarzschild vacuum solution at the radius, R, where the pressure becomes zero. The radius, R, is determined by inserting (Q_*, S_*, C_*) into the expression for r in (14). This leads to

$$R = \sqrt{\frac{\eta(1-Q_*^2)}{8\pi\rho_0(Q_*^2 - S_*^2)}}\ . \tag{26}$$

From $Q_*^2 - S_*^2 - C_* = 0$ and $C_* > 0$, it follows that $Q_*^2 \neq S_*^2$. Hence all spherically symmetric solutions are finite.

2.1. The boundary structure

To understand the structure of the interior state space, one has to understand the structure of the boundary subsets. Orbits belonging to the scale-invariant ($Q = 1$) boundary are shown in figure 2a,b while orbits belonging to the plane-symmetric ($C = 0$) boundary are shown in figure 2c,d. We refrain from explicitly showing orbits for models with η in the interval $\sqrt{64/7} - 2 < \eta < 2$, since these models only require a slight modification of figure 2a. In addition, the incompressible fluid case $\eta = 1$ (which belongs to the shown $1 \leq \eta \leq \sqrt{64/7} - 2$ interval), and the causal fluid cases $\eta \geq 2$, are the most interesting ones. We require the solutions to have non-negative pressure and, as seen from the above discussion, the surface of vanishing pressure cut all solutions before they come into the neighborhood of the $W = 0$ boundary. Thus this boundary plays a physically less important role than the other boundary subsets and we therefore refrain from showing orbits belonging to this subset.

3. REGULAR SOLUTIONS

Rather than considering all possible orbits, and the associated solutions' physical features, we focus on orbits corresponding to solutions with regular centers and positive pressure, the so-called *regular subset* of solutions.

The spacetime for a regular star is described by the flat Minkowski geometry at the center. Hence, all orbits belonging to the regular subset start from the equilibrium point P_3^+, which corresponds to Minkowski spacetime on spherically symmetric form. In the vicinity of P_3^+, one finds the following approximate expression for the regular orbits[4]

$$Q = 1 - \epsilon_1 \mathrm{e}^{2\lambda} , \tag{27a}$$
$$S = \tfrac{2}{3} (\epsilon_2 - \epsilon_1) \mathrm{e}^{2\lambda} , \tag{27b}$$
$$C = 1 - \tfrac{4}{2+\eta} \epsilon_2 \mathrm{e}^{2\lambda} , \tag{27c}$$

where ϵ_1 and ϵ_2 are small positive constants. It is, however, only the quotient $0 \leq \epsilon_1/\epsilon_2 \leq +\infty$ that parameterizes the 1-parameter subset of regular solutions, which can be seen from

$$\lim_{\lambda \to -\infty} \frac{p}{\rho} = \frac{p_c}{\rho_c} = \frac{\rho_c - \rho_0}{(\eta - 1)\rho_c} = \frac{2 - (2+\eta)(\epsilon_1/\epsilon_2)}{2(\eta - 1) + (2+\eta)(\epsilon_1/\epsilon_2)} , \tag{28}$$

[4]These expressions are found from the eigenvectors associated with the two positive eigenvalues (both equal to 2) of the equilibrium point P_3^+.

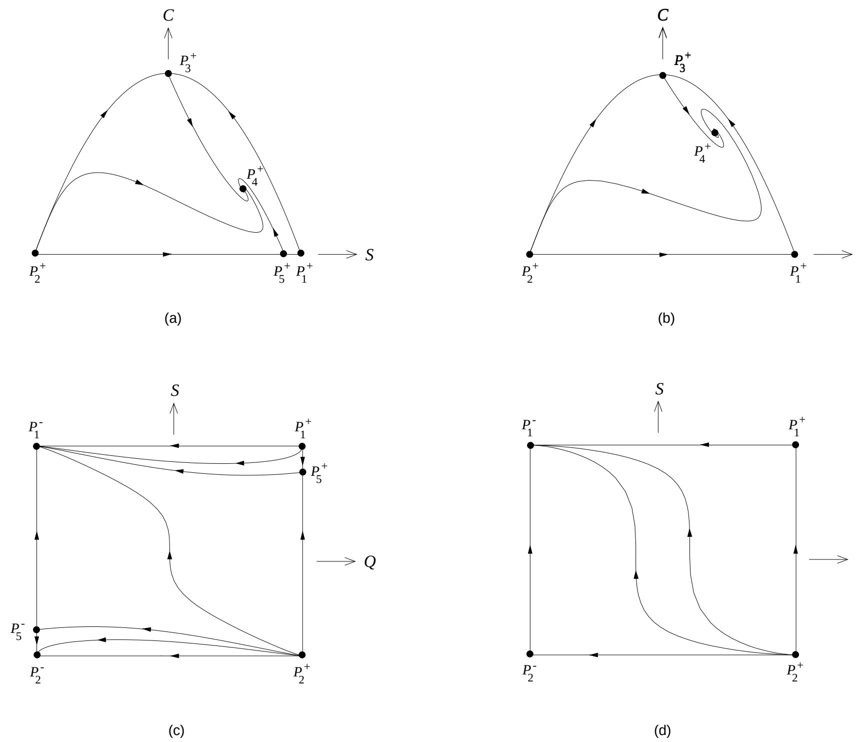

FIG. 2. Orbits belonging to the boundary subsets for static spherically symmetric models with a linear equation of state, using the variables $\{Q, S, C\}$, with (a) the scale-invariant boundary $Q = 1$ with $1 \leq \eta \leq \sqrt{64/7} - 2$, (b) the scale-invariant boundary $Q = 1$ with $\eta > 2$, (c) the plane-symmetric boundary $C = 0$ with $\eta < 2$, and (d) the plane-symmetric boundary $C = 0$ with $\eta \geq 2$.

where p_c and ρ_c denote the values of the pressure and energy density at the center of the star. From (28) we see that there exists solutions with a regular center but negative pressure. For solutions with non-negative pressure at the center, the quotient ϵ_1/ϵ_2 is subject to the constraint

$$\frac{\epsilon_1}{\epsilon_2} \leq \frac{2}{2+\eta} \ , \tag{29}$$

where the equality corresponds to a vanishing central pressure. Setting $\epsilon_1 = 0$ corresponds to an eigendirection in the $Q = 1$ subset and is associated with taking the limit $p_c \to \infty$ for the interior solutions. The quotient p_c/ρ_c is a gravitational strength parameter. The Newtonian limit corresponds to small values of this parameter. Relativistic effects are thus most

pronounced when this parameter takes as large values as possible. The maximal value $\frac{p_c}{\rho_c} = \frac{1}{\eta - 1}$, describing the high pressure limit, is obtained when $\epsilon_1 = 0$, which corresponds to the $Q = 1$ subset. The $Q = 1$ subset can thus be expected to play a role when probing relativistic effects.

Solutions describing stars correspond to orbits that start from the equilibrium point P_3^+ and satisfy the positive pressure criteria in (29). These orbits eventually pass through the surface of vanishing pressure at an interior point (Q_*, S_*, C_*) of the state space. From (26) it follows that the linear equation of state with $\rho_0 \neq 0$ leads to star models with finite radii. This is expected, since star models always are finite when $\rho \neq 0$ as $p \to 0$ (see Rendall & Schmidt [11]).

The qualitative behavior of the regular subset, projected onto the plane-symmetric boundary ($C = 0$) is shown in figure 3a for $1 \leq \eta \leq \sqrt{64/7} - 2$ and in figure 3b for models with $\eta \geq 2$. The intersection between the orbits in the regular subset and the surface of vanishing pressure is indicated by the dashed lines. Two orbits constitute the "high pressure" boundary of the regular subset. The first orbit belongs to the boundary $Q = 1$ and starts at P_3^+ and ends at P_4^+. This orbit corresponds to a unique regular scale-invariant solution with infinite radius, in contrast to models with $\rho_0 \neq 0$ (see, for example, Collins [3]). Thus solutions with high central pressure are approximately described by this solution near their center. The other orbit enters the interior state space from the "Tolman" equilibrium point P_4^+. This orbit represents a non-regular solution, but nevertheless plays an important role for regular models, particularly those with high central energy density and pressure. We will refer to this orbit as the Tolman orbit (although a corresponding exact solution is known only for the "stiff" fluid case, $\eta = 2$, see Tolman [15]).

For the incompressible fluid case and low values of η, this orbit forms the boundary of a "simple" surface formed by the regular subset. For larger values of η ($\eta > \sqrt{64/7} - 2$), the self-similar Tolman point P_4^+ is a focus. This leads to a more complicated situation where "high central pressure" orbits in the regular subset spiral around the Tolman orbit, which in this case also acts as a "skeleton" orbit for the regular subset. Thus the "regular boundary" orbit, the Tolman point P_4^+, and the Tolman orbit are seen to play key roles for the regular subset. Moreover, it is possible to obtain approximations for solutions with high central pressure by piecewise joining perturbations of the regular boundary orbit and the Tolman orbit.

In the interior of the star, a mass function $m(r)$ can be defined (see, for example, Misner & Sharp [8]). The quotient $m(r)/r$ is dimensionless, and can be written as

$$\frac{m}{r} = \frac{C - (Q - S)^2}{2C} \ . \tag{30}$$

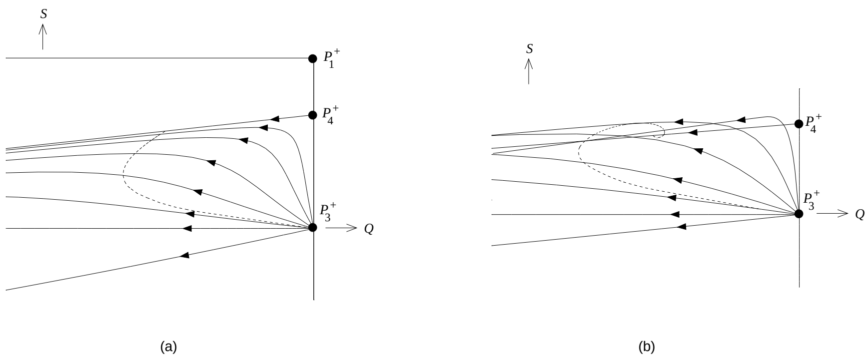

FIG. 3. Orbits belonging to the regular subset for models with a linear equation of state, using the variables $\{Q, S, C\}$, projected onto the plane-symmetric boundary $C = 0$ for the cases (a) $1 \le \eta \le \sqrt{64/7} - 2$, and (b) $\eta \ge 2$. The intersection of the regular subset and the surface of vanishing pressure is indicated by the dotted lines.

Equation (26) and (30) imply that the total mass M of the star is given by

$$M = \frac{S_* \sqrt{\eta(1 - Q_*^2)}}{(Q_* + S_*)\sqrt{8\pi\rho_0(Q_*^2 - S_*^2)}} \ . \tag{31}$$

This constant is to be identified with the mass parameter of the Schwarzschild solution (5) when an interior solution is matched with the exterior Schwarzschild solution.

Besides the orbits of the regular subset and the single orbit starting from the equilibrium point P_4^+, there is also a 2-parameter set of orbits starting from the point P_2^+ and a 1-parameter set of orbits starting from P_5^+ (when $\eta < 2$). These two latter sets of orbits, however, describe solutions that start out with negative mass. Positive mass is subsequently added and the solutions eventually aquire a total positive mass at a sufficiently large radius.

4. COMMENTS ON POSSIBLE GENERALIZATIONS OF THE EQUATION STATE

In this Section we address the issue whether it is possible to modify the above formulation so as to be useful for studying models with other barotropic equations of state. As will be seen in this series of papers, the behavior when $\rho, p \to \infty$ and when $p \to 0$ constitute key ingredients when one attempts to find useful formulations for a given equation of state. The latter limit naturally leads to two types of equations of state; those for

which $\rho \to 0$ when $p \to 0$ and those for which $\rho \to \rho_0 > 0$ when $p \to 0$. As shown by Rendall & Schmidt [11], the latter case always leads to models with finite radii. In the first case the situation is more complicated and one sometimes have models with finite radii and sometimes not.

The present formulation can be modified to cover the situation when $\rho \to \rho_0 > 0, asp \to 0$, and for some equations of state this may lead to a useful approach. Let us consider the following class of equations of state:

$$\rho = \rho_0 + [\eta(p/\rho_0) - 1]p \ , \tag{32}$$

where $\rho_0 > 0$ and where η is an analytic function satisfying $1 \leq \eta < \infty$, for all non-negative values of the pressure. Thus, η is viewed as function and not a constant. Equation (21) yields

$$(p/\rho_0)\eta(p/\rho_0) = \frac{Q^2 - S^2 - C}{1 - Q^2} \ , \tag{33}$$

and hence

$$\eta = \eta\left(\frac{Q^2 - S^2 - C}{1 - Q^2}\right) \ . \tag{34}$$

To obtain a dynamical system, one first writes the given equation of state in the form (32). Then one uses (33) to directly or indirectly determine the function η in terms of the variables Q, S and C. Finally one introduces $\eta(\frac{Q^2-S^2-C}{1-Q^2})$ into the dynamical system (17a)-(17c), possibly changing the independent variable in a suitable way.

This formulation, however, cannot be used when $\rho \to 0, p \to 0$, which happens for, *e.g.*, polytropic equations of state. In a sequel to this paper we will discuss other types of formulations, suitable for treating such problems. Moreover, one can also use these formulations when $\rho \to \rho_0 > 0, p \to 0$ and treat models with for example, linear equations of state. The present formulation has some advantages in this case, however, and it also sheds light on these other formulations. It may also be useful to combine the various approaches for certain equations of state.

5. CONCLUSION

We have expressed the field equations as a 3-dimensional regular system on a compact state space for static spherically symmetric models with a linear equation of state. In this formulation, the existence of regular solutions is trivial since they all start from a hyperbolic equilibrium point. The fact that all models of this type have finite radii has been naturally incorporated into the formalism. We have shown that all models are asymptotically

self-similar for small radii. We have obtained a global picture of the solution space and this has revealed that certain solutions, the regular scale-invariant solution, the self-similar Tolman solution, and the non-regular solution associated with the central infinite pressure limit, play key roles for understanding the solution structure, and the structure of the regular solutions with large central pressure in particular.

These special solutions exist in all models with linear equations of state, including the incompressible fluid case, and basically play the same role, although there are some differences depending on if the Tolman equilibrium point is a focus or not. For the incompressible fluid, these solutions are intimately connected with the Buchdahl inequalities (one of the inequalities is an equality for the solution corresponding to the incompressible Tolman orbit), see, for example, Buchdahl [1] and Hartle [6]. In a sequel to this paper we will show that similar key solutions exist for other equations of state as well. Thus many of the features encountered in this paper are typical for large classes of equations of state.

An interesting application of formulations of this type, for a given equation of state, is to probe how physical features depend on, for example, the central pressure. In addition, the understanding of the solution space for a class of equation of state allows one to investigate how different physical features, like, for example, stability properties, depend on the equation of state. Another possible application is perturbation theory. Since one has a good understanding about the background solutions, one might investigate how details of the equation of state affect the possible gravitational wave forms

ACKNOWLEDGMENT

This research was supported by Gålöstiftelsen (USN), Svenska Institutet (USN), Stiftelsen Blanceflor (USN), the University of Waterloo (USN), and the Swedish Natural Research Council (CU).

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